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A UNIFIED THEORY FOR CORRELATING STEADY LAMINAR NATURAL CONVECTIVE HEAT TRANSFER DATA FOR HORIZONTAL ANNULI*

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NOMENCLATURE

a, $= (n+1)/3$;
Gr_x, Grashof number, $\beta g T^* x^3/v^2$; see equation (5);
m, $= 2\,(n-1/2)$;
Nu, Nusselt number $[\partial T/\partial y]|_{y=0}\,\Delta'/(T_i'-T_o')$;
n, a dimensionless constant which describes the thermal boundary conditions;
P, local dynamic pressure;
Pr, Prandtl number, $v\alpha^{-1}$;
R, characteristic radius for the Rayleigh No. [see equations (7) and (8)];
Ra_R, $= Gr_R Pr$;
T, local temperature;
u, local velocity in the principal flow direction;
v, local velocity transverse to the principal flow direction;
$(\ )'$, a constant dimensional quantity.

Greek symbols
α, thermal diffusivity;
β, thermal coefficient of volumetric expansivity;
v, kinematic viscosity;
χ, local similarity variable;
ψ, stream function, see equation (2).

Subscripts
I, intermediate region;
i, inner region;
o, outer region.

INTRODUCTION

EXPERIMENTAL and theoretical research has been devoted to the study of natural convective heat transfer in horizontal annuli for at least the last half century [1–8]. Many different techniques for correlating the mean heat transfer have been proposed [e.g. 1, 2].

Accurate correlations for natural convective heat transfer in annuli with regular (e.g. concentric circular cylinders) and irregular (e.g. hexagonal cylinder inside a circular cylinder) boundaries is becoming increasingly important in many technological areas. For example, much research is being conducted in the development of shipping casks which are used to passively cool spent nuclear reactor fuel subassemblies [6, 7]. The existence of such correlations can reduce or eliminate the need for experimentation for a particular application.

Previous investigators [e.g. 1–3] have shown, via dimensional analysis of the governing equations, that the mean Nusselt number or heat transfer can be correlated as a function of the Rayleigh number, Prandtl number and a geometry or aspect ratio. Most of the fundamental considerations stopped at this point. As a result, the characteristic length and temperature for both the Nusselt and Rayleigh numbers were arbitrarily chosen. Itoh, Nishiwaki and Hirata [1] realized that the characteristic lengths of the Nusselt and Rayleigh number must be different. However, they determined such lengths by assuming that the thermal energy transport was due to thermal conduction rather than the coupled mechanisms of natural convection and thermal conduction. The present theory is based on fundamental concepts and a detailed formulation of the various flow regimes which exist in the annulus. This theory clearly specifies the characteristic quantities and their relation to the aspect ratio (Δ'/r'). Although the results which are presented here are for isothermal concentric circular cylinders (regular boundaries), there appears to be the capability to extend the technique to annuli with irregular boundaries.

THEORY (SUMMARY)

Consider a horizontal annulus of arbitrary cross-section with wall temperature distributions for the inner and outer boundaries of the form $T_i' x_i^{m_i}$ and $T_o' x_o^{m_o}$ respectively (see Fig. 1). For the sake of brevity and clarity, it is assumed that $T_i > T_o$.

The steady two-dimensional annulus flow is divided into six distinct flow regions: (1) inner boundary layers (thermal and momentum), (2) outer boundary layers, (3) inner intermediate, (4) outer intermediate, (5) plume, and (6) stably stratified. The boundary layer regions are perhaps the most important for data correlation and the easiest to understand in the classical sense. That is, the boundary layer regions are characterized by strong lateral diffusion and comparable convection. The boundary layers are formed near the annulus boundaries and are caused by the destabilizing local temperature gradients in the gravity field. The plume region is located adjacent to the axis of symmetry and above the inner boundary where flow separation occurs. Mass flows (via the plume region) from the inner boundary layer to the outer boundary layer.

The intermediate region has several important characterics. It is characterized by a flow reversal (i.e. $u_{Ii} = 0 = u_{Io}$) which is caused by recirculation of fluid in the annulus. The locus of points at which this reversal occurs identifies the boundary between the inner and outer intermediate regions. In addition, there is a significant thermal stratification in the x_I-direction. Therefore, thermal diffusion is significant in the x_I-direction but is negligible in the y_I-direction. Finally, since the intermediate region flow occurs outside the boundary layers, that flow is inviscid.

A complete formulation of the governing equations, associated boundary conditions, and matching conditions are

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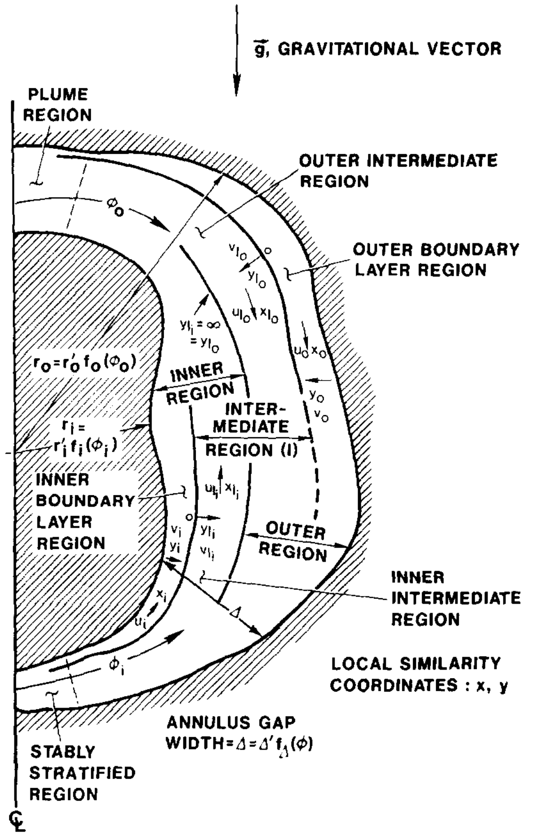

FIG. 1. A schematic of the steady two-dimensional laminar natural convective flow regions existing in a horizontal annulus.

necessary in order to completely describe this annulus flow. Although this has been done, it is beyond the scope of this communication. It is unnecessary to solve such equations completely to obtain an appropriate form for the correlation parameters. However, such solutions would be useful to characterize new geometries and boundary conditions for physical problems in the absence of experimental data.

The governing equations for the boundary layer regions are

$$\frac{\partial u}{\partial x} + \frac{\partial y}{\partial y} = 0$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\rho^{-1}\frac{\partial P}{\partial x} + v\frac{\partial^2 u}{\partial y^2} \mp g\beta\,[T - T_I(y_I = 0)]\sin\phi$$

$$-\frac{u^2}{r} = -\rho^{-1}\frac{\partial P}{\partial y} + g\beta[T - T_I(y_I = 0)]\cos\phi. \quad (1)$$

The minus and plus signs in the third equation apply to the outer and inner boundary layers, respectively. Curvature effects have been neglected.

The following local similarity transformations, using coordinate stretching variables, were obtained

$$\chi = Gr_x^{a/2}\, y x^{-1}$$

$$T - T_I(y_I = 0) = T^* Gr_x^{(2a-1)}\,\Theta(\chi)$$

$$\psi = vGr_x^{a/2}\, F(\chi),$$

and

$$P = \rho v^2 Gr_x^{2a} x^{-2} G(\chi), \quad (2)†$$

where

$$u = \frac{\partial\psi}{\partial y},$$

$$v = -\frac{\partial\psi}{\partial x},$$

$$a = (n+1)/3,$$

$$m = 2(n - 1/2)$$

$$Gr_x = \beta g T^* x^3 v^{-2},$$

$$T_i^* = T'_i - T'_{Ii}$$

and

$$T_o^* = T'_{Io} - T'_o.$$

In the above equations, unsubscripted quantities such as Gr_x, n (or a) and T^* apply to both the inner and outer boundary layer regions. Further, T'_{Ii} and T'_{Io} are constant characteristic temperatures in the inner and outer intermediate regions, respectively.

For steady two-dimensional flow in a horizontal annulus, the mean heat transferred from the inner boundary must equal the mean heat transferred to the outer boundary, i.e.

$$\overline{\left.\frac{\partial T_i}{\partial y_i}\right|_{y_i=0}} = -\overline{\left.\frac{\partial T_o}{\partial y_o}\right|_{y_o=0}},$$

or

$$(x_{i_{max}} - x_i^*)^{-1}\int_{x_i^*}^{x_{i_{max}}}\left.\frac{\partial T_i}{\partial y_i}\right|_{y_i=0} dx_i = -(x_{o_{max}} - x_o^*)^{-1}\int_{x_o^*}^{x_{o_{max}}}\left.\frac{\partial T_o}{\partial y_o}\right|_{y_o=0} dx_o \quad (3)$$

where x^* (corresponding to $\phi = 0$) is determined by matching, e.g. the inner boundary layer region to the stably stratified region. The quantity $x_{i_{max}}$ corresponds to $\phi = \pi$. It has been assumed that the spatial extent of the plume and stably stratified regions is sufficiently small that the above integrals are approximated adequately using the formulation results from the boundary layer region in equation (3). After the boundary layer transformation is substituted into equation (3) and the results simplified, it is found that

$$T'_i - T'_{Ii} = (T'_o - T'_{Io})^{a_o/a_i}\,\xi(\beta g v^{-2})^{a_o/a_i - 1} \quad (4a)$$

where

$$\xi^{5/2\,a_i} = C_1 C_2 \frac{(r'_o)^{5n_o/2-3/2}}{(r'_i)^{5n_i/2-3/2}}$$

and

$$C_1 = \int_0^\pi f_i(\phi_i)\,d\phi_i\left[\int_0^\pi f_o(\phi_o)\,d\phi_o\right]^{-1} \quad (4b)$$

$$C_2 = -\left[\int_0^\pi \Theta_{o\chi}(0) f_o(\phi_o)\left(\int_0^{\phi_o} f_o(\phi_o)\,d\phi_o\right)^{5n_o/2-3/2} d\phi_o\right] \times \left[\int_0^\pi \Theta_{i\chi}(0) f_i(\phi_i)\left(\int_0^{\phi_i} f_i(\phi_i)\,d\phi_i\right)^{5n_i/2-3/2} d\phi_i\right]^{-1}. \quad (4c)$$

As noted earlier, since the temperature gradients, with respect to y_I, are small in the intermediate region, $T'_{Ii} = T'_{Io} = T'_I$. Also note that when the inner and outer thermal boundary conditions are of the same form (e.g. isothermal) $a_i = a_o = a$ (or $n_i = n_o = n$) so that the above become

$$T'_I = (1+\xi)^{-1}(T'_i + \xi T'_o), \quad T_i^* = (1+\xi^{-1})^{-1}(T'_i - T'_o)$$

† Differentiation of dimensionless functions such as Θ, F and G with respect to χ is denoted by e.g. Θ_χ, F_χ and G_χ, respectively.

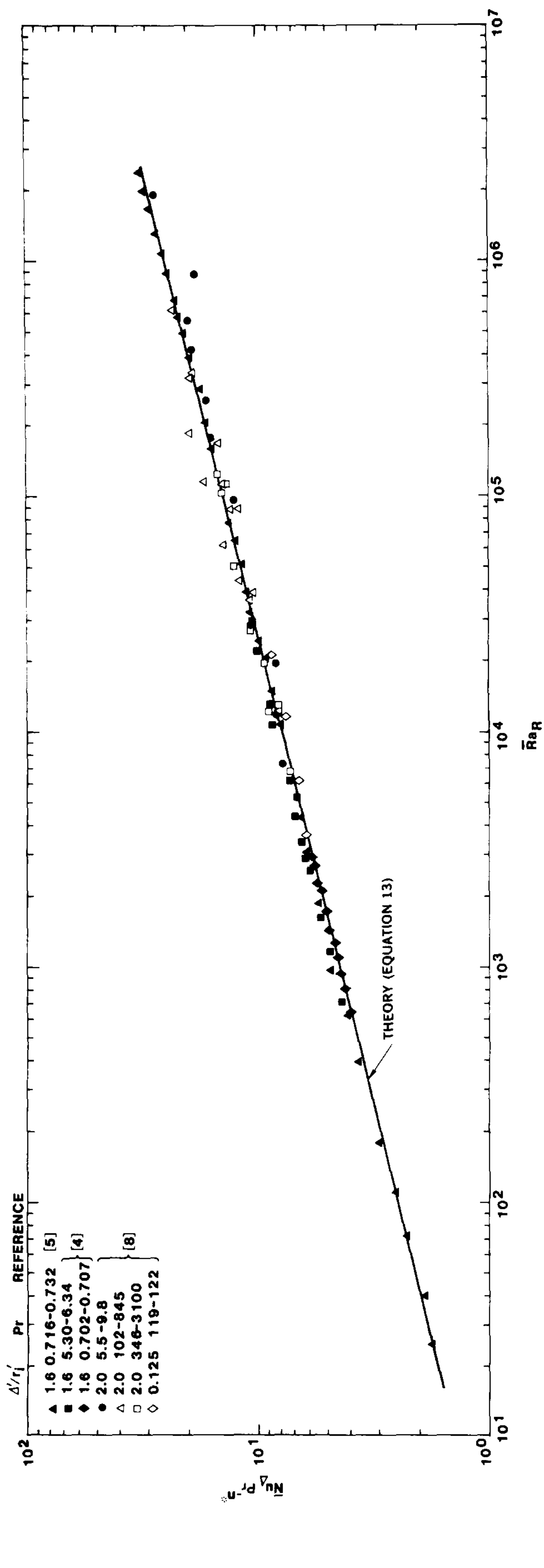

FIG. 2. Presentation of the mean natural convective heat transfer data, for isothermal concentric circular cylinders, in terms of the new correlation variables identified by the unified correlation theory. The data is from Kraussold [5] and Kuehn and Goldstein [3, 4].

and

$$T_o^* = (1+\xi)^{-1}(T_i' - T_o'). \tag{5}$$

In identifying the appropriate correlation relationship, the mean Nusselt number is defined as

$$\overline{Nu_\Delta} \equiv \left.\frac{\overline{\partial T}}{\partial y}\right|_{y=0} \frac{\Delta'}{(T_i' - T_o')}. \tag{6}$$

With this definition for Nu_Δ, no additional arbitrary choices will be made for a reference temperature or length for the Grashof (Gr) or Rayleigh (Ra) number. Using the similarity transformation for T_i, we find that the correlation for the laminar natural convective mean heat transfer in a horizontal annulus is

$$\overline{Nu_{\Delta_i}} = C_3\, Pr^{(1-5n_i)/6}\, Ra_{R_i}^{(5n_i-1)/6} = \overline{Nu_{\Delta_o}}, \tag{7}$$

where

$$Ra_{R_i} = Pr\, Gr_{R_i} = \beta g \alpha^{-1} \nu^{-1} T_i^* R_i^3$$

$$R_i^3 = (1+\xi^{-1})^{6/(1-5n_i)} (r_i')^{3(5n_i-3)/(5n_i-1)} (\Delta')^{6/(5n_i-1)}$$

and

$$C_3 = \left[\int_0^\pi f_i(\phi_i)\, d\phi_i\right]^{-1} \times \int_0^\pi \left[\int_0^{\phi_i} f_i(\phi_i)\, d\phi_i\right]^{5n_i/2-1/2} \Theta_{i\chi}(0)\, d\phi_i.$$

The Prandtl number (Pr) was included so that the effect of different fluids could be included explicitly in the correlation.

RESULTS AND DSCUSSION

Due to the lack of experimental data for irregular annuli, the correlation defined in equations (4)–(7) has been evaluated for the case of isothermal concentric cylinders. From equation (7), the correlation for the mean Nusselt number in an annulus formed by isothermal ($n_i = n_o = 1/2$) concentric circular cylinders ($C_1 = 1 = C_2$) is

$$\overline{\mathrm{Nu}_\Delta} = C_3\, Pr^{-1/4}\, Ra_{R_i}^{1/4} \tag{8}$$

where

$$R_i^3 = (1+\xi^{-1})^{-4} (r_i')^{-1} (\Delta')^4, \quad \xi = \left(\frac{r_o'}{r_i'}\right)^{-1/5}$$

and

$$C_3 = \pi^{-1} \int_0^\pi \phi_i^{3/4}\, \Theta_{i\chi}(0)\, d\phi_i.$$

Notice that C_3 is a function of Prandtl number (Pr) through its dependence on $\Theta_{i\chi}(0)$. Since Pr can vary significantly for different fluids, equation (8) was rewritten in a form which makes the least squares data reduction simpler (e.g. see [8])

$$\overline{Nu_\Delta} = C_4\, Pr^{n^*}\, Ra_{R_i}^{1/4} \tag{9}$$

where

$$n^* = C_5 + C_6\, Pr^{-1/3}$$

and C_4, C_5 and C_6 are constants.

The versatility of the correlation technique was demonstrated using the steady two-dimensional laminar mean heat transfer measurements made by Kraussold [5], and Kuehn and Goldstein [3, 4]. The result (Fig. 2) of the least squares fit is

$$\overline{Nu_\Delta} = 0.796\, Pr^{n^*}\, Ra_{R_i}^{1/4} \tag{10}$$

where

$$n^* = 0.00663 - 0.0351\, Pr^{-1/3}$$

$10^1 < Ra_R < 10^7$, $0.706 \le Pr \le 3100$, and $0.125 \le \Delta'/r_i' \le 2.0$ and Δ'/r_i' is the aspect ratio for the annulus. It is interesting to note that $Pr^{n^*} = 0.9975 \pm 2\%$ (nearly 1.0) for all values of Pr included in the correlation. Since curvature effects are not included, this correlation is not expected to be valid for $\Delta'/r' > 3.0$.

These results, which display the ability of the present correlation technique to correlate experimental data over a large parameter range, are very encouraging. Even with these large parameter variations, the present correlation technique collapses all the experimental data to a *single line*, which is given by equation (10) for the case of isothermal concentric cylinders. Previous correlations have resulted in a family of curves with aspect ratio as a parameter, due to the arbitrary selection of a characteristic length. Based on the work completed to date, the physical problem appears to be completely specified by equation (7) when the following is known: boundary conditions (i.e. n), the fluid (i.e. Pr), the aspect ratio (Δ'/r_i'), and the Grashof or Rayleigh number.

This correlation technique was recently found to apply to the irregular annulus formed by an inner hex and an outer concentric circular cylinder. This approach is unique from previous ones in that it demonstrates that *a priori* choices of the reference temperature and length can be made for either the Nusselt number or the Rayleigh number but not for both. The definition of remaining reference quantities is uniquely determined by the former choice.

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